Indian Statistical Institute, Bangalore
M. Math First Year
Second Semester - Differential Geometry I

Final Exam — Date: May 09, 2018
Maximum Marks: — Duration: 3 hours

Attempt all questions. Each question carries 15 marks.

1. (a) Let $M := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1 + z^2\}$ be the hyperboloid of one-sheet in R^3. Let $U = M \cap \{y = 0, x > 0\}$. Using suitable local coordinates on U, compute the 1st fundamental form on U.
   (b) Compute the 2nd fundamental form of M above, and its mean and Gaussian Curvatures.

2. (a) Show that the curve $\alpha(t) = (3t - t^3, 3t^2, 3t + t^3)$ is not a helix but that torsion equals the curvature at each point.
   (b) Define a geodesic on a surface. Prove that the geodesics on a sphere are great circles.

3. (a) Let $\sigma(u, v) = (u + v, u - v, uv)$ be a parametrization of a surface s. Calculate the Gaussian and mean curvatures.
   (b) Let $S = \{(x, y, z)|z > 0, x^2 + y^2 + z^2 = 1\}$. Define and find the principal curvatures of S and geodesic curvature of the curve $S \cap \{z = 1/2\}$.
   (c) Let $g : U \to \mathbb{R}$ be a smooth function defined on an open set $U \subseteq \mathbb{R}^2$. Define $\sigma : U \to \mathbb{R}^3$ by $\sigma(u, v) = (u, v, g(u, v))$. Show that Area $(\sigma) = \int\limits_U [1 + (\frac{\partial g}{\partial u})^2 + (\frac{\partial g}{\partial v})^2]^{\frac{1}{2}}$.

4. (a) Give the definitions of length of a curve on a surface, isometry between surfaces. Is the map from the cone $x^2 + y^2 = z^2, z > 0$ to the plane given by $(x, y, z) \to (x, y, 0)$ an isometry?
   (b) When is a map between surfaces conformal? Let $f(x)$ be a smooth function. Let $\sigma(u, v) = (u \cos v, u \sin v, f(u))$ be the parametrization of the surface of revolution S obtained by rotating the curve $z = f(x)$ in the $xz-$ plane about the z-axis. Find all the functions f for which σ is conformal.